Weak Non Linearity and Magneto-Acoustic Waves

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We show that the property of a linear system viz, the fact that square of the amplitude propagates with the group velocity continues to be valid in the case of magneto acoustic waves in the presence of weak non linearity.

The fact that for slowly varying wave trains in a linear system the square of the amplitude propagates with the group velocity is well known. In presence of weak non linearity, the behaviour of wave trains for different system has been studied by several authors [1, 2]. Shivamoggi [2] has shown that in the case of weakly linear longitudinal waves in a hot plasma the above mentioned result of linear systems holds, whereas his method of analysis fails for ion acoustic waves. It is then natural to ask whether this property continues to hold in the case of magneto-acoustic waves. As our analysis shows, the answer is in the affirmative.

Consider the magnetic acoustic waves propagating across a magnetic field in a cold collision free plasma [3]. In this system the field quantities are reduced to (i) ion density N (ii) ion fluid velocity $\boldsymbol{V}$ (iii) the magnetic field $\boldsymbol{B}$ [4].

Restricting the system to one dimensional plane waves propagating along the x direction and taking the applied magnetic field to be along the z direction we have

$$\boldsymbol{V} = (u, v, 0)\,, \tag{1}$$

$$\boldsymbol{B} = (0, 0, B_z)\,. \tag{2}$$

The basic system of equations can be written in the normalized form [4]

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\,u) = 0\,, \tag{3}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{n}\frac{\partial}{\partial x}\left(\frac{B_z^2}{2}\right) = 0\,, \tag{4}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{R_\mathrm{e}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\frac{\partial B_z}{\partial x}\right) = 0\,, \tag{5}$$

$$\frac{\mathrm{d}B_z}{\mathrm{d}t} + B_z\frac{\partial u}{\partial z} = -\frac{1}{R_i}\frac{\partial}{\partial x}\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right), \tag{6}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{7}$$

and where n has been normalized by the unperturbed density N_0, the ion fluid velocity v by the Alfven speed $B_0/[4\pi(m_\mathrm{i}+m_\mathrm{e})\,N_0]^{1/2}$ and B by the strength of the applied magnetic field B_0. m_i and m_e denote the masses of the ions and the electrons, R_i and R_e are the normalized ion and electron Larmor frequencies, respectively.

Equations (3) to (6) can be written as

$$\frac{\partial U}{\partial t} + v_1\frac{\partial U}{\partial x} + v_2\frac{\partial}{\partial x}\left[I\frac{\partial}{\partial t} + UI\frac{\partial}{\partial x}\right]U = 0\,, \tag{8}$$

where

$$U = \begin{bmatrix} n \\ u \\ v \\ B_z \end{bmatrix}, \tag{9}$$

$$v_1 = \begin{bmatrix} u & n & 0 & 0 \\ 0 & u & 0 & B_{z/n} \\ 0 & 0 & u & 0 \\ 0 & B_z & 0 & u \end{bmatrix}, \tag{10}$$

$$v_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_\mathrm{e}^{-1}\quad n^{-1} \\ 0 & 0 & R_i^{-1} & 0 \end{bmatrix}, \tag{11}$$

I is the 4×4 unit matrix and

$$R_\mathrm{i} = (m_\mathrm{e}/m_\mathrm{i})^{1/2}\,, \quad R_\mathrm{e} = (m_\mathrm{i}/m_\mathrm{e})^{1/2}\,. \tag{12}$$

To solve the system of equations (8) we expand all quantities to represent slowly varying wave trains of the form

$$B_z = 1 + \varepsilon B_{z1}(\zeta, \tau, \theta) + \varepsilon^2 B_{z2}(\zeta, \tau, \theta) + \ldots\,, \tag{13}$$

$$n = 1 + \varepsilon\, n_1(\zeta, \tau, \theta) + \varepsilon^2 n_2(\zeta, \tau, \theta) + \ldots\,, \tag{14}$$

$$u = \varepsilon\, u_1(\zeta, \tau, \theta) + \varepsilon^2 u_2(\zeta, \tau, \theta) + \ldots\,, \tag{15}$$

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where

$$\zeta = \varepsilon x\,, \quad \tau = \varepsilon t\,, \tag{16}$$

$$\theta = k(\zeta, \tau)\, x - w(\zeta, \tau)\, t\,. \tag{17}$$

From the first set of equations for $O(\varepsilon)$, we have the starting solutions as follows:

$$\begin{aligned} B_{z1} &= a(\zeta, \tau)\, e^{i\theta} + \bar{a}(\zeta, \tau)\, e^{-i\theta}\,, \\ n_1 &= \frac{k^2}{w^2}[a(\zeta, \tau)\, e^{i\theta} + \bar{a}(\zeta, \tau)\, e^{-i\theta}]\,, \\ u_1 &= \frac{k}{w}[a(\zeta, \tau)\, e^{i\theta} + \bar{a}(\zeta, \tau)\, e^{-i\theta}]\,, \\ v_1 &= -\frac{ik}{R_e}[a(\zeta, \tau)\, e^{i\theta} - \bar{a}(\zeta, \tau)\, e^{-i\theta}]\,, \end{aligned} \tag{18}$$

where the bar denotes the complex conjugate and the dispersion relation takes the form

$$k^2 - w^2 = k^2 w^2\,. \tag{19}$$

The second set of equations for $O(\varepsilon^2)$ are given by

$$\frac{\partial n_2}{\partial t} + \frac{\partial n_1}{\partial \tau} + \frac{\partial u_1}{\partial \zeta} + \frac{\partial u_2}{\partial x} + \frac{\partial}{\partial x}(n_1 u_1) = 0\,, \tag{20}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial B_{z1}}{\partial \zeta} + \frac{\partial B_{z2}}{\partial x} + B_{z1}\frac{\partial B_{z1}}{\partial x} - n_1 \frac{\partial B_{z1}}{\partial x} = 0\,, \tag{21}$$

$$\begin{aligned} &\frac{\partial v_2}{\partial t} + \frac{\partial v_1}{\partial \tau} + u_1 \frac{\partial v_1}{\partial x} + \frac{1}{R_e}\frac{\partial B_{z2}}{\partial x\, \partial t} + \frac{1}{R_e} u_1 \frac{\partial B_{z1}}{\partial x} \frac{-kw}{R_e}(n_1 B_{z1}) + \frac{1}{R_e} \\ &\quad \cdot (ik a_\tau - k a \theta_\tau - i w_\zeta a - i w a_\zeta + w a \theta_\zeta)\, e^{i\theta} = 0\,, \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{\partial B_{z2}}{\partial t} + \frac{\partial B_{z1}}{\partial \tau} + B_{z1}\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial \zeta} + u_1 \frac{\partial B_{z1}}{\partial x} + \frac{1}{R_i}\frac{\partial^2 v_2}{\partial x\, \partial t} + \frac{1}{R_i}\frac{\partial}{\partial x}\left(u_1 \frac{\partial v_1}{\partial x}\right) \\ &\quad + (k k_\tau a + k^2 a_\tau + i k^2 a \theta_C - k_\zeta w a - k w_\zeta a - k w a_\zeta - i k w a \theta_\zeta)\, e^{i\theta} + \text{c.c.} = 0\,. \end{aligned} \tag{23}$$

Using (18) and (19), (20)–(23) become

$$\begin{aligned} &\left(\frac{\partial^2 B_{z2}}{\partial t^2} - \frac{\partial^2 B_{z2}}{\partial x^2} - \frac{\partial^4 B_{z2}}{\partial x^2 \partial t^2}\right) + \frac{\partial^2}{\delta x\, \delta t}(k w n_1 B_{z1}) - \frac{1}{R_i}\frac{\delta^2}{\delta x\, \delta t}\left(u_1 \frac{\delta V_1}{\delta x}\right) \\ &- \frac{\delta^2}{\partial x\, \partial t}\left(u_1 \frac{\partial B_{z1}}{\partial x}\right) + \frac{\partial}{\partial t}\left(B_{z1}\frac{\partial u_1}{\partial x}\right) + \frac{\partial}{\partial t}\left(u_1 \frac{\partial B_{z1}}{\partial x}\right) + \frac{1}{R_i}\frac{\delta^2}{\delta t\, \delta x}\left(u_1 \frac{\delta V_1}{\delta x}\right) \\ &- \frac{\delta}{\delta x}\left(u_1 \frac{\delta u_1}{\delta x}\right) - \frac{\partial}{\partial x}\left(B_{z1}\frac{\partial B_{Z1}}{\partial x}\right) + \frac{\partial}{\partial x}\left(n_1 \frac{\partial B_{z1}}{\partial x}\right) + \left[-\frac{2k^2}{w} a_\tau + 2k(w^2 - 1)\, a_\zeta + 3 k w a w_\zeta\right] e^{i\theta} \\ &+ \left[-\frac{2k^2}{w}\bar{a}_\tau + 2k(w^2 - 1)\, \bar{a}_\zeta + 3 k w \bar{a} w_\zeta\right] e^{-i\theta} = 0\,. \end{aligned} \tag{24}$$

By removing the secular terms in (13), we have,

$$\begin{aligned} &-\frac{2k^2}{w} a_\tau + 2k(w^2 - 1)\, a_\zeta + 3 k w a w_\zeta = 0\,, \\ &-\frac{2k^2}{w}\bar{a}_\tau + 2k(w^2 - 1)\, \bar{a}_\zeta + 3 k w \bar{a} w_\zeta = 0\,. \end{aligned} \tag{25}$$

Using the dispersion relation (19), (25) becomes

$$\bar{a} a_\tau + \bar{a}_\tau a + \frac{w^3}{k^3}(\bar{a} a_\zeta + \bar{a}_\zeta a) - \frac{3 w^2}{k} a \bar{a} w_\zeta = 0\,,$$

which can be written in the forms

$$\frac{\partial}{\partial \tau}[|A|^2] + \frac{\partial}{\partial \zeta}[C(k)\, |A|^2] = 0\,, \tag{26}$$

$$C(k) = \frac{\mathrm{d}w}{\mathrm{d}k} = \frac{w^3}{k^3}\,. \tag{27}$$

Equation (27) shows that the quantity $|A|^2$ propagates with the group velocity.

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